Perturbations of quadratic centers of genus one

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Abstract

We propose a program for finding the cyclicity of period annuli of quadratic systems with centers of genus one. As a first step, we classify all such systems and determine the essential one-parameter quadratic perturbations which produce the maximal number of limit cycles. We compute the associated Poincaré-Pontryagin-Melnikov functions whose zeros control the number of limit cycles. To illustrate our approach, we determine the cyclicity of the annuli of two particular reversible systems.

0. Introduction

As well known, there are four types of planar quadratic systems with a center: 1) Hamiltonian, 2) reversible, 3) generalized Lotka-Volterra, and 4) of codimension four. The first integral in the Hamiltonian case is a cubic polynomial in (x, y) whose generic level sets are elliptic curves. It is easy to observe that the generic level sets of the first integral of the codimension four case are elliptic curves too (see the end of Section 4). Our main efforts will be devoted to the remaining two cases. It has been recently proved by one of the authors (S.G.) that there are 18 classes of reversible centers, 6 classes of reversible Lotka-Volterra centers and 5 classes of generic (non-reversible) Lotka-Volterra centers whose phase portraits contain only elliptic curves (possibly, a finite number of them reducible) [5]. They are given by codimension-one or codimension-two algebraic sets in the space of all centers from the corresponding type, see Theorems 1 and 2 below. Throughout the paper, by "genus" we mean the genus of the compactified and normalized generic phase curves. An algebraic phase curve is generic if it does not contain a singular point of the vector field in its closure.

The centers whose (generic complexified) periodic orbits are elliptic curves will be called *centers of genus one*. We note that even a quadratic system can have a center of arbitrarily big genus.

Once we know that the first integral of a given planar system with a center defines elliptic curves, we could raise the next question: how many limit cycles could be produced in the phase portrait under small quadratic (or even polynomial) perturbations? *The purpose of the paper is to present a program for solving this problem.* In what follows we restrict our attention to the limit cycles which bifurcate from open period annuli (we do not consider graphics). It turns out that, instead of multi-parameter perturbations, it is enough to consider suitable *one-parameter* small quadratic perturbations, see [9]. The one-parameter perturbations which produce the maximal number of limit cycles in the quadratic case, called *essential perturbations*, together with the corresponding generating functions of limit cycles (or also Poincaré-Pontryagin-Melnikov functions) were determined in [14].

The first part of our program is to adapt the results of [14] to our case. The result is a complete list of such essential perturbations of quadratic systems with centers of genus one, together with the corresponding generating functions. The list is presented in Section 3 (in the reversible case) and in Section 4 (in the Lotka-Volterra case).

The second part of the program is to study the zeros of the generating functions $I(t)$ found in Sections 3 and 4. The fact that each $I(t) = \int_{\{H=t\}} \omega$ is a complete elliptic integral from a rational one-form ω, over the level sets $\{H = t\} \subset \mathbb{C}^2$ which are elliptic curves, allows one to apply all related facts from algebraic geometry in order to estimate the number of zeros of $I(t)$ and thus to set up some upper (or lower) bounds on the number of bifurcating limit cycles in the system, see e.g. [20, 19, 11, 7, 8]. This part of our program is illustrated in Section 5, where we study the cyclicity of the period annuli for two types of quadratic reversible systems with centers of genus one.

The paper is organized as follows. In Section 1 we determine all reversible centers with phase portrait formed by elliptic curves. In Section 2 we discuss the same question for the Lotka-Volterra centers (reversible or not). These results were previously proved in [5]. For convenience of the reader we present here almost self-contained proofs adapted for the purposes of the present paper. In Sections 3 and 4 we determine for each of the cases the corresponding generating function, the complete elliptic integral $I(t)$ which is the leading term in the expansion of the first return mapping, respectively for the reversible and Lotka-Volterra cases. We also present there (as Conjectures 1 and 2) the expected exact upper bounds for the number of zeros of all generating functions. In Section 5 we use the geometric properties of the elliptic fibration determined by the first integral of the quadratic system, in order to determine, for two of the cases, the number of the zeros of $I(t)$. From this we deduce an exact result: the cyclicity of the annuli under consideration is two (Theorem 3).

1. The reversible case

The general first integral of a reversible system with a center at the origin

$$\dot{z} = -iz + az^2 + 2|z|^2 + b\bar{z}^2, \quad a, b \in \mathbb{R}, \quad z = x + iy \tag{1.1}$$

is given by

$$H(x, y) = X^\lambda(\tfrac{1}{2}y^2 + AX^2 + BX + C) \tag{1.2}$$

where $X = 1+2(a-b)x$ and λ, A, B, C are explicit rational functions of the parameters a, b (see formula (1.7) below). Moreover, one has $\lambda \neq 0, -1, -2$. Along with the elliptic Hamiltonian $H = \frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{4}{3}x^3$ (corresponding to $a = b = -1$ in (1.1)), formula (1.2) contains all the cases for which $\{H = t\}$ is (possibly) an algebraic curve of genus 1. Below, we are going to determine all these cases.

We begin with the observation that it is enough to consider only the case when $\lambda < -1$. Indeed, if $\lambda > -1$, by the bi-rational change $(X, y) = (1/X_1, Y_1/X_1)$ the function (1) reduces to $H_1 = X_1^{-2-\lambda}(\frac{1}{2}Y_1^2 + CX_1^2 + BX_1 + A)$ with $-2 - \lambda < -1$. For this reason below we will consider the case when

$$\lambda = -\frac{p}{q} \quad \text{with} \quad p, q \in \mathbb{N}, \ \ p > q, \ \ p \neq 2q, \ \ gcd(p, q) = 1$$

where $gcd(p, q)$ denotes the greatest common divisor of p and q.

1. Assume first that $A \neq 0$, $C \neq 0$. Then $\{H = t\}$ after taking $X = x^q$ (this is an isomorphic map [5], Lemma 1) reads

$$\tfrac{1}{2}y^2 = -Ax^{2q} - Bx^q - C + tx^p. \tag{1.3}$$

For arbitrary t, the curve (1.3) has a genus 1 if and only if the polynomial on the right hand side has a degree 3 or 4. That is, either $p = 3$ and $q = 1, 2$ or $p = 4$ and $q = 1$.

Consider now the degenerate cases when either $A = 0$ or $C = 0$ (let us note that neither two of the coefficients A, B, C can vanish simultaneously).

2. Assume that $A = 0$. Then (1.3) is of genus 1 if and only if either $p = 3$, $q = 1, 2$ or $p = 4$, $q = 1, 3$. Thus the unique new case compared to the case $A \neq 0$ is $(p, q) = (4, 3)$.

3. Let $C = 0$. If $2q > p > q$, by a bi-rational transformation $(x, y) = (1/x_1, y_1/x_1^q)$ one can transform (1.3) into

$$\tfrac{1}{2}y^2 = -A - Bx^q + tx^{2q-p} \tag{1.4}$$

(here and below, we shall omit the subscript 1). As $q > 2q - p$, the genus is 1 if and only if either $q = 3$, $p = 4, 5$ or $q = 4$, $p = 5, 7$. If $p > 2q$ and q is even, then the rational change $y = y_1x^{q/2}$ in (1.3) yields

$$\tfrac{1}{2}y^2 = -Ax^q - B + tx^{p-q}. \tag{1.5}$$

As $p-q>q$, (1.5) is of genus 1 if and only if $(p,q)=(5,2)$. Finally, if $p>2q$ and q is odd, the rational change $y=y_1x^{(q-1)/2}$ in (1.3) yields

$$\tfrac{1}{2}y^2=-Ax^{q+1}-Bx+tx^{p-q+1}. \tag{1.6}$$

As above, (1.6) is of genus one if and only if $p-q+1$ equals 3 or 4, which is possible when either $(p,q)=(3,1)$ or $(p,q)=(4,1)$, cases that already have been obtained when we assumed that $A\neq 0$.

Thus we have obtained the complete list of cases for which $\{H=t\}$, H given by (1.2) and t arbitrary, is a curve of genus one (the right column contains the cases with $\lambda>-1$):

(0) $H=\frac{1}{2}y^2+Ax^2+Bx^3$	(the standard elliptic case)
(i) $H=X^{-3}(\frac{1}{2}y^2+AX^2+BX+C)$	(ii) $H=X(\frac{1}{2}y^2+CX^2+BX+A)$
(iii) $H=X^{-3/2}(\frac{1}{2}y^2+AX^2+BX+C)$	(iv) $H=X^{-1/2}(\frac{1}{2}y^2+CX^2+BX+A)$
(v) $H=X^{-4}(\frac{1}{2}y^2+AX^2+BX+C)$	(vi) $H=X^{2}(\frac{1}{2}y^2+CX^2+BX+A)$
(vii) $H=X^{-4/3}(\frac{1}{2}y^2+BX+C)$	(viii) $H=X^{-2/3}(\frac{1}{2}y^2+CX^2+BX)$
(ix) $H=X^{-4/3}(\frac{1}{2}y^2+AX^2+BX)$	(x) $H=X^{-2/3}(\frac{1}{2}y^2+BX+A)$
(xi) $H=X^{-5/3}(\frac{1}{2}y^2+AX^2+BX)$	(xii) $H=X^{-1/3}(\frac{1}{2}y^2+BX+A)$
(xiii) $H=X^{-5/4}(\frac{1}{2}y^2+AX^2+BX)$	(xiv) $H=X^{-3/4}(\frac{1}{2}y^2+BX+A)$
(xv) $H=X^{-7/4}(\frac{1}{2}y^2+AX^2+BX)$	(xvi) $H=X^{-1/4}(\frac{1}{2}y^2+BX+A)$
(xvii) $H=X^{-5/2}(\frac{1}{2}y^2+AX^2+BX)$	(xviii) $H=X^{1/2}(\frac{1}{2}y^2+BX+A)$.

We should mention that in cases (iii) and (iv) above it was assumed that $C\neq 0$. If $C=0$, the curve has a genus zero, see below.

Let us now recall the exact formula of (1.2) from [14]:

$$H(X,y)=X^{-\frac{a+b+2}{a-b}}\left(\frac{y^2}{2}+\frac{1}{8(a-b)^2}\left(\frac{a+b-2}{a-3b-2}X^2+2\frac{b-1}{b+1}X+\frac{a-3b+2}{a+b+2}\right)\right). \tag{1.7}$$

This formula holds outside the lines $a=b$, $a+b+2=0$, $b=-1$, $a-3b-2=0$ (note the last three cases correspond to $\lambda=0,-1,-2$ in (1.2)). Except for the three points $(a,b)=(-1,-1),(\pm 2,0)$, on these lines the first integrals contain exponents [14] and hence their level sets are not algebraic curves. For $(a,b)=(\pm 2,0)$ the level sets are conic ovals. The lines $a+b+2=-\lambda(a-b)$ in the (a,b)-plane, $\lambda\in\mathbb{R}$, together with $a=b$, form the bundle of straight lines through the point $(-1,-1)$ which corresponds to the standard elliptic case. Therefore using the above results, we obtain:

Theorem 1. *The phase curves of* (1.1) *are algebraic curves of genus one if and only*

if one of the conditions holds:

(r1) $a = 2b + 1$	(r2) $a = -1$ (the reversible Hamiltonian case)
(r3) $a = 5b + 4$ $(b \neq -3)$	(r4) $a = -3b - 4$ $(b \neq -3)$
(r5) $a = \frac{5}{3}b + \frac{2}{3}$	(r6) $a = \frac{1}{3}b - \frac{2}{3}$
(r7) $(a, b) = (\frac{5}{2}, -\frac{1}{2})$	(r8) $(a, b) = (-\frac{7}{2}, -\frac{1}{2})$
(r9) $(a, b) = (-8, -2)$	(r10) $(a, b) = (4, -2)$
(r11) $(a, b) = (-17, -5)$	(r12) $(a, b) = (7, -5)$
(r13) $(a, b) = (-7, -\frac{5}{3})$	(r14) $(a, b) = (\frac{11}{3}, -\frac{5}{3})$
(r15) $(a, b) = (-23, -7)$	(r16) $(a, b) = (9, -7)$
(r17) $(a, b) = (13, 5)$	(r18) $(a, b) = (-3, 5)$.

For completeness, below we add the list of all cases in (1.1) for which the ovals $\{H = t\}$ are conic curves (ellipses). These are

(r19) $H = X^{-3/2}(\frac{1}{2}y^2 + AX^2 + BX)$ when $(a, b) = (-11, -3)$

(r20) $H = X^{-1/2}(\frac{1}{2}y^2 + BX + A)$ when $(a, b) = (5, -3)$

(r21) $H = X^{-2}(\frac{1}{2}y^2 + BX + C)$ when $(a, b) = (2, 0)$

(r22) $H = \frac{1}{2}y^2 + CX^2 + BX$ when $(a, b) = (-2, 0)$.

Cases (r21) and (r22) (not included in (1.2) and (1.7)) are taken from the full list of first integrals of (1.1), see e.g. [14]. The cases (r19) and (r20) are obtained from (1.2) in the same way as above. Note that (r20) and (r21) are the isochronous centers $\mathcal{S}_3$ and $\mathcal{S}_2$, respectively. By the way, the isochronous center $\mathcal{S}_4$ corresponds to $b = 1$ in (r5). Perturbations of the quadratic isochronous centers have been studied in [3].

2. The Lotka-Volterra case

A (generalized) Lotka-Volterra system with a center at the origin has in complex coordinates the form

$$\dot{z} = -iz + Az^2 + B\bar{z}^2, \quad z, A, B \in \mathbb{C}. \tag{2.1}$$

Apart of the classical Lotka-Volterra system, the generalized one splits into real and complex cases. In appropriate coordinates, the general first integral of the Lotka-Volterra system in the classical *real case* is

$$H(x, y) = x^\lambda y^\mu (1 - x - y), \quad \lambda, \mu \in \mathbb{R}, \quad \lambda\mu(\lambda + \mu + 1) \neq 0. \tag{2.2}$$

The general first integral in the *complex case* is

$$H(x, y) = (x^2 + y^2)^\lambda (1 - 2x) e^{-2\mu \operatorname{Arctan}(y/x)}, \quad \lambda, \mu \in \mathbb{R}, \quad \lambda < -\frac{1}{2}. \tag{2.3}$$

When $\mu = 0$ in the complex case and when $(\lambda - \mu)(\lambda - 1)(\mu - 1) = 0$ in the real case, the corresponding system becomes reversible. That is, after a suitable affine

change of the variables, the initial system with a first integral (2.2) or (2.3) will take in complex coordinates one of the normal forms

$$\dot{z} = -iz + z^2 + b\bar{z}^2, \quad b \in \mathbb{R}; \quad \dot{z} = -iz + \bar{z}^2. \tag{2.4}$$

This is possible provided that the coefficients in (2.1) satisfy $A^3B \in \mathbb{R}$. Our main result in this section is the following.

Theorem 2. *The phase curves of* (2.1) *are algebraic curves of genus one if and only if one of the conditions holds:*

$$\begin{array}{ll}
\text{(rlv1) } A = 0 & \text{(the Hamiltonian triangle)} \\
\text{(rlv2) } 2AB - \bar{A}^2 = 0 & \text{(lv1) } AB + (1 \pm 2i)\bar{A}^2 = 0 \\
\text{(rlv3) } AB - 3\bar{A}^2 = 0 & \text{(lv2) } 169AB - (101 \pm 28i)\bar{A}^2 = 0 \\
\text{(rlv4) } 5AB - 3\bar{A}^2 = 0 & \text{(lv3) } 289AB - (151 \pm 42i)\bar{A}^2 = 0 \\
\text{(rlv5) } 5AB - \bar{A}^2 = 0 & \text{(lv4) } 1681AB - (783 \pm 60\sqrt{2}i)\bar{A}^2 = 0 \\
\text{(rlv6) } 3AB + \bar{A}^2 = 0 & \text{(lv5) } 841AB - (349 \pm 12i)\bar{A}^2 = 0
\end{array}$$

The above statement is a consequence of the following two propositions which will be proved together:

Proposition 1. *The phase curves of the reversible Lotka-Volterra system* (2.1), *$A^3B \in \mathbb{R}$, are algebraic curves of genus one if and only if its first integral is affine equivalent to one of the normal forms*

$$\begin{array}{ll}
\text{(rlv1) } H = xy(1 - x - y) & \text{(the Hamiltonian triangle)} \\
\text{(rlv2) } H = x^{-3}y(1 - x - y) & \\
\text{(rlv3) } H = x^2y(1 - x - y) & \\
\text{(rlv4) } H = x^{-4}y(1 - x - y) & \\
\text{(rlv5) } H = (x^2 + y^2)^{-\frac{2}{3}}(1 - 2x) & \\
\text{(rlv6) } H = (x^2 + y^2)^{-2}(1 - 2x). &
\end{array}$$

Proposition 2. *The phase curves of the generic Lotka-Volterra system* (2.1), *$A^3B \notin \mathbb{R}$, are algebraic curves of genus one if and only if the first integral is affine equivalent to one of the normal forms*

$$\begin{array}{l}
\text{(lv1) } H = x^2y^3(1 - x - y) \\
\text{(lv2) } H = x^{-6}y^2(1 - x - y) \\
\text{(lv3) } H = x^{-6}y^3(1 - x - y) \\
\text{(lv4) } H = x^{-4}y^2(1 - x - y) \\
\text{(lv5) } H = x^{-3}y^{\frac{3}{2}}(1 - x - y).
\end{array}$$

Proof of Propositions 1 and 2. Under the conditions in (2.3), the critical point $\left(\frac{\lambda}{2\lambda+1}, \frac{\mu}{2\lambda+1}\right)$ is a center. The origin is a focus for $\mu \neq 0$ and a center elsewhere. Clearly, the phase curves defined by (2.3) could be elliptic only if $\mu = 0$. Note that if $\mu = 0$, the origin is a center for all $\lambda \neq 0$, but this center is reversible and the system can be transformed into the normal form (1.1). Here we study the Lotka-Volterra center outside the origin, existing for $\lambda < -\frac{1}{2}$.

Under the conditions in (2.2) (frankly, one should take $H = xy(1-x-y)|x|^{\lambda-1}|y|^{\mu-1}$ there, but modules will be omitted thoroughly), there is a unique critical point $\left(\frac{\lambda}{\lambda+\mu+1}, \frac{\mu}{\lambda+\mu+1}\right)$ outside the invariant straight lines $x = 0$, $y = 0$, $x + y = 1$, which is a center if and only if $\lambda\mu(\lambda+\mu+1) > 0$. In fact, there are two topologically different configurations having a center, the first one is obtained for $\lambda > 0, \mu > 0$ and the second one corresponds to the parameters outside the first quadrant. In this latter case, one can take without loss of generality $\lambda < 0$, $\mu < 0$. Indeed, if e.g. $\lambda < 0 < \mu$, then applying an affine change $x = 1 - X - Y$, $y = Y$, we reduce the first integral $H(x, y) = t$ in (2.2) to $H_1(X, Y) = t_1$ where

$$H_1(X, Y) = X^{1/\lambda} Y^{\mu/\lambda}(1 - X - Y), \quad t_1 = t^{1/\lambda} \tag{2.5}$$

One can proceed similarly with the other case $\mu < 0 < \lambda$. In the same way, when λ and μ are positive, we can reduce their values to $\lambda, \mu \in (0, 1]$. Indeed, if e.g. $\lambda > 1$ and $\lambda \geq \mu$, the same change as above transforms (2.2) into (2.5), with both degrees in (2.5) within $(0, 1]$. We proceed similarly if $\mu > 1$ and $\mu \geq \lambda$.

When $\lambda\mu(\lambda + \mu + 1) = 0$, the first integral in (2.2) should be replaced by another one containing logarithmic or exponential terms [14, 21, 23] and therefore its level sets are not algebraic curves. In this way we have reduced our consideration to the following two cases

(I) $0 < \lambda \leq 1$, $0 < \mu \leq 1$,

(II) $\lambda < 0$, $\mu < 0$, $\lambda + \mu + 1 > 0$.

In the reversible Lotka-Volterra cases, one can find all curves with a genus one by simply using the results from the previous section. Take the real reversible case and assume for definiteness that $\lambda = \mu$. The remaining possibilities are reduced to this one by the same change which we used to obtain (2.5) from (2.2). Let $\lambda = \mu \in (-1/2, 1]$. A further substitution in (2.2) $x = \frac{1}{2}(1 - X) + Y$, $y = \frac{1}{2}(1 - X) - Y$ transforms the curve $H(x, y) = t$ to $H_1(X, Y) = t_1$ where

$$H_1(X, Y) = X^{1/\lambda}[-\tfrac{1}{2}Y^2 + \tfrac{1}{8}(X - 1)^2], \quad t_1 = \tfrac{1}{2}t^{1/\lambda}. \tag{2.6}$$

As $1/\lambda \in (-\infty, -2) \cup [1, \infty)$ and (2.6) takes the form already studied in Section 1 above, we conclude immediately that H_1, and hence H, is of genus one if and only if $1/\lambda \in \{-4, -3, 1, 2\}$. Namely, for $\lambda = \mu \in \{1, \frac{1}{2}, -\frac{1}{4}, -\frac{1}{3}\}$ in (2.2). In the same way, taking in the complex case (2.3) $\mu = 0$ and using the substitution $x = \frac{1}{2}(1 - X)$, $y = Y$, we reduce (2.3) to (2.6) (with a sign $+$ in front of $\frac{1}{2}$). As $1/\lambda \in (-2, 0)$ now, one obtains a curve of genus one if and only if $1/\lambda \in \{-\frac{3}{2}, -\frac{1}{2}\}$. That is, for $\lambda \in \{-\frac{2}{3}, -2\}$ and $\mu = 0$ in (2.3). Thus we have obtained the complete list of cases

with genus one in the reversible Lotka-Volterra system (the first three entries in the right column contain the cases with $\mu = 1$; the cases with $\lambda = 1$ are obtained by a rotation of the variables $(x, y) \to (y, x)$)

$$
\begin{array}{ll}
H = xy(1-x-y) & \\
H = x^{\frac{1}{2}}y^{\frac{1}{2}}(1-x-y) & H = x^2y(1-x-y) \\
H = x^{-\frac{1}{4}}y^{-\frac{1}{4}}(1-x-y) & H = x^{-4}y(1-x-y) \\
H = x^{-\frac{1}{3}}y^{-\frac{1}{3}}(1-x-y) & H = x^{-3}y(1-x-y) \\
H = (x^2+y^2)^{-\frac{2}{3}}(1-2x) & H = (x^2+y^2)^{-2}(1-2x).
\end{array}
$$

For a completeness, let us add to the above list the unique reversible Lotka-Volterra case having conic orbits, namely

$$\text{(rlv0)}\ H = (x^2+y^2)^{-1}(1-2x).$$

This is the quadratic isochronous center known as $\mathcal{S}_1$ and it corresponds to $b = 0$ in (2.4). This finishes the proof of Proposition 1.

Now we come to the generic (non-reversible) real cases (I), (II) above, where $(\lambda - \mu)(\lambda - 1)(\mu - 1) \neq 0$. Consider first (I). Let

$$\lambda = \frac{p}{q}, \quad \mu = \frac{r}{s}, \quad p, q, r, s \in \mathbb{N}, \quad p < q, \quad r < s, \quad gcd(p, q) = 1, \quad gcd(r, s) = 1.$$

We replace our first integral $H(x, y) = t$ with $H_1(x, y) = t_1$ where:

$$H_1(x, y) = x^{p_1}y^{q_1}(1-x-y)^{r_1}, \quad p_1 = \lambda r_1, q_1 = \mu r_1, t_1 = t^{r_1}, \quad r_1 = qs/gcd(q, s). \tag{2.7}$$

The irreducible algebraic curve

$$\Gamma_t = \{(x, y) \in \mathbb{C}^2 : x^p y^q (1-x-y)^r = t\}, \quad p, q, r \in \mathbb{N}, \quad p < q < r, \quad gcd(p, q, r) = 1 \tag{2.8}$$

is smooth if and only if $t \notin \Delta = \{0, (\frac{r}{p+q+r})^{\frac{p^2+q^2+r^2}{r}}\}$. Let $\bar{\Gamma}_t$ be the associated compact Riemann surface. Its (geometric) genus is one and the same for all $t \notin \Delta$. We have (cf. [5, 6]):

Proposition 3. *The compact Riemann surface $\bar{\Gamma}_t$, $t \notin \Delta$, is of genus one if and only if $p = 1$, $q = 2$, $r = 3$.*

Proof. For reader's convenience, we repeat the proof from [5]. It is based on the Poincaré-Hopf formula, namely $deg(\omega) = 2g - 2$, which we apply to the meromorphic one-form related to the first integral (2.8)

$$\omega = -\frac{dx}{x[q - qx - (q+r)y]} = \frac{dy}{y[p - py - (p+r)x]}.$$

We recall that, in our context, $deg(\omega) = \sum_k n_k$ where the summation is taken over all zeros or poles of ω, and n_k are their orders. The above one-form has neither zeroes

nor poles in the affine chart outside the critical locus. Hence it suffices to consider it at infinity. There are 3 points at infinity: $S_1 = [1 : 0 : 0]$, $S_2 = [0 : 1 : 0]$, $S_3 = [1 : -1 : 0]$. The local coordinates near S_1 can be chosen as follows. Write $x = 1/u$ with $u \to 0$. After this change of coordinates, equation (2.8) becomes $y^q(1-u+yu)^r = u^{p+r}$ (we take $t = (-1)^r$ for simplicity). Since $u \to 0$, we have $y \to 0$ (if $y \to \infty$, we would have $yu \to -1$ which corresponds to S_3). Consequently, up to bi-analyticity, there are local coordinates (u, y) verifying $y^q = u^{p+r}$. Let $m = gcd(q, p + r)$. There are m different parameterizations, namely:

$$u = \xi^{\frac{q}{m}}, \quad y = e^{\frac{2ik\pi}{m}} \xi^{\frac{p+r}{m}}, \quad k = 0, \dots, m-1,$$

which correspond to the m different local coordinates near the m smooth points given after blowing-up S_1. In these local coordinates,

$$\omega = \frac{-ud(1/u)}{q - q/u - (q+r)y} = \frac{qd\xi}{m\xi[q - q\xi^{-\frac{q}{m}} - (q+r)e^{\frac{2ik\pi}{m}}\xi^{\frac{p+r}{m}}]}.$$

We conclude that near the m points, ω has a zero of order $\frac{q}{m} - 1$.

We perform the same study near the two other points at infinity. Near S_2, by symmetry, we obtain $n = gcd(p, q + r)$ points where the one-form ω has a zero of order $\frac{p}{n} - 1$. Similarly, near S_3, there are $l = gcd(r, p + q)$ points where ω has a zero of order $\frac{r}{l} - 1$.

As a result, we obtain the formula $deg(\omega) = p + q + r - n - m - l$ and therefore the curve (2.8) is elliptic if and only if

$$p + q + r = n + m + l. \tag{2.9}$$

Now we have to resolve this Diophantine equation. Obviously, we have $m \leq q$, $n \leq p$ and $l \leq r$, hence (2.9) is true if and only if

$$gcd(q, p + r) = q, \quad gcd(p, q + r) = p, \quad gcd(r, p + q) = r.$$

Take natural numbers α, β, γ such that $q + r = p\alpha, \quad p + r = q\beta, \quad p + q = r\gamma$. The latter system has a nonzero solution

$$\frac{p}{r} = \frac{\gamma+1}{\alpha+1}, \quad \frac{q}{r} = \frac{\gamma+1}{\beta+1}$$

if and only if

$$\alpha\beta\gamma = 2 + \alpha + \beta + \gamma.$$

By (2.8) we have $\alpha > \beta > \gamma$ and the unique solution satisfying this condition is $\alpha = 5$, $\beta = 2$, $\gamma = 1$ which leads to $p/r = 1/3$, $q/r = 2/3$. $\square$

To consider the second case (II), we note that a bi-rational change of the variables

$$x = \frac{-X}{1 - X - Y}, \quad y = \frac{-Y}{1 - X - Y}$$

transforms (II) to a case already considered: $H_1 = X^\Lambda Y^M(1 - X - Y) = t_1$ with $\Lambda > 0$, $M > 0$, where

$$\Lambda = \frac{-\lambda}{\lambda + \mu + 1}, \quad M = \frac{-\mu}{\lambda + \mu + 1}, \quad t_1 = t^{-1/(\lambda+\mu+1)}. \tag{2.10}$$

Hence one can use the values of Λ, M already obtained above to calculate λ and μ for case (II) through the formula

$$\lambda = \frac{-\Lambda}{\Lambda + M + 1}, \quad \mu = \frac{-M}{\Lambda + M + 1}.$$

Thus we have completed the list of all cases with genus one in the generic Lotka-Volterra system (another 15 cases are obtained by a rotation of the variables):

$$\begin{array}{lll}
H = x^{\frac{2}{3}}y^{\frac{1}{3}}(1-x-y) & H = x^{\frac{3}{2}}y^{\frac{1}{2}}(1-x-y) & H = x^2y^3(1-x-y) \\
H = x^{-\frac{1}{6}}y^{-\frac{1}{3}}(1-x-y) & H = x^{-6}y^2(1-x-y) & H = x^{\frac{1}{2}}y^{-3}(1-x-y) \\
H = x^{-\frac{1}{6}}y^{-\frac{1}{2}}(1-x-y) & H = x^{-6}y^3(1-x-y) & H = x^{\frac{1}{3}}y^{-2}(1-x-y) \\
H = x^{-\frac{1}{4}}y^{-\frac{1}{2}}(1-x-y) & H = x^{-4}y^2(1-x-y) & H = x^{\frac{1}{2}}y^{-2}(1-x-y) \\
H = x^{-\frac{1}{3}}y^{-\frac{1}{2}}(1-x-y) & H = x^{-3}y^{\frac{3}{2}}(1-x-y) & H = x^{\frac{2}{3}}y^{-2}(1-x-y).
\end{array}$$

Proposition 2 is completely proved. □

Proof of Theorem 2. The proof is a direct consequence of Propositions 1 and 2. More precisely, it follows by a straightforward calculation using the formulas from [14], case (iv) on page 157 there. □

3. The generating function in the reversible case

We are going to study small perturbations of the reversible system (1.1) in the cases when the first integral H defines a curve of genus one. Consider first the *quadratic* perturbations of system (1.1) rewritten in real coordinates:

$$\begin{aligned}
\dot{x} &= H_y/M + \varepsilon f(x, y, \varepsilon), \\
\dot{y} &= -H_x/M + \varepsilon g(x, y, \varepsilon)
\end{aligned} \tag{3.1}$$

where $M = X^{\lambda-1}$ and f, g are quadratic polynomials with coefficients depending analytically on the small parameter ε. Given a perturbation (f, g), the limit cycles in (3.1) are determined by the zeroes of the leading term $I(t)$ in the expansion with respect to ε of the displacement map. For this reason the integral $I(t)$ is called sometimes the *generating function*. As well known [14], one can always choose a particular quadratic perturbation so that the corresponding $I(t)$ would have the possible maximum of zeroes within the whole class of quadratic perturbations. In the generic case $I(t)$ is given by the complete elliptic integral [14], Theorems 2 and 3

$$I(t) = \int_{\delta(t)} x^{\lambda-1}(\mu_1 + \mu_2 x + \mu_3 x^{-1})y dx \tag{3.2}$$

where $\delta(t)$ is the oval contained in the level set $H = t$ and $\mu_i \in \mathbb{R}$. In the exceptional cases (r10) and (r5) with $b = 2$ when system (1.1) belongs to the intersection with the codimension-four stratum Q_4 of the center manifold (see [23] for details), $I(t)$ takes another form and is not an Abelian integral [14]. In the standard elliptic case when $a = b = -1$ in (1.1) (we will denote it by (r0)), the integral has a specific form, too. Using (3.2) and the list of first integrals of genus one (0)-(xviii) above, we obtain the concrete form of $I(t)$ for all the cases (after appropriate re-scaling of x, y, t, H, I) as follows:

(r0) $I(t) = \int_{H=0} (\mu_1 + \mu_2 t + \mu_3 x) y dx,$
$H = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3 - t, \;\; t \in (-\frac{1}{6}, 0)$

(r1) $I(t) = \int_{H=0} x^{-4}(\mu_1 + \mu_2 x + \mu_3 x^{-1}) y dx,$
(r2) $I(t) = \int_{H=0} x^{-3}(\mu_1 + \mu_2 x^{-1} + \mu_3 x) y dx,$
$H = \frac{1}{2}y^2 + \frac{3-b}{6(b+1)} + \frac{b-1}{b+1}x + \frac{1-3b}{2(b+1)}x^2 - tx^3$

(r3) $I(t) = \int_{H=0} x^{-4}(\mu_1 + \mu_2 x^2 + \mu_3 x^{-2}) y dx,$
(r4) $I(t) = \int_{H=0} x^{-2}(\mu_1 + \mu_2 x^{-2} + \mu_3 x^2) y dx,$
$H = \frac{1}{2}y^2 + \frac{b+3}{24(b+1)} + \frac{b-1}{4(b+1)}x^2 + \frac{3b+1}{8(b+1)}x^4 - tx^3$

(r5) $I(t) = \int_{H=0} x^{-5}(\mu_1 + \mu_2 x + \mu_3 x^{-1}) y dx, \;\; (b \neq 2)$
(r5) $I(t) = \int_{H=0} x^{-5}[\mu_1 + \mu_2 x + \mu_3 x^{-2} + \mu_4(1 - x^{-1}) \ln x] y dx, \;\; (b = 2)$
(r6) $I(t) = \int_{H=0} x^{-4}(\mu_1 + \mu_2 x^{-1} + \mu_3 x) y dx,$
$H = \frac{1}{2}y^2 + \frac{2-b}{4(b+1)} + \frac{b-1}{b+1}x + \frac{1-2b}{2(b+1)}x^2 - tx^4$

(r7) $I(t) = \int_{H=0} x^{-5}(\mu_1 + \mu_2 x^3 + \mu_3 x^{-3}) y dx,$
(r8) $I(t) = \int_{H=0} x^{-2}(\mu_1 + \mu_2 x^{-3} + \mu_3 x^3) y dx,$
$H = \frac{1}{2}y^2 + \frac{1}{12} - \frac{1}{3}x^3 - tx^4, \;\; t \in (-\frac{1}{4}, 0)$

(r9) $I(t) = \int_{H=0} (\mu_1 + \mu_2 x^{-3} + \mu_3 x^3) y dx,$
(r10) $I(t) = \int_{H=0} x^{-3}[\mu_1 + \mu_2 x^3 + \mu_3 x^6 + \mu_4(2x^3 - 1 - x^{-3}) \ln x] y dx,$
$H = \frac{1}{2}y^2 + \frac{1}{6} + \frac{1}{3}x^3 - tx^2, \;\; t \in (\frac{1}{2}, \infty)$

(r11) $I(t) = \int_{H=0} x(\mu_1 + \mu_2 x^{-3} + \mu_3 x^3) y dx,$
(r12) $I(t) = \int_{H=0} x^{-2}(\mu_1 + \mu_2 x^3 + \mu_3 x^{-3}) y dx,$
$H = \frac{1}{2}y^2 + \frac{1}{3} + \frac{1}{6}x^3 - tx, \;\; t \in (\frac{1}{2}, \infty)$

(r13) $I(t) = \int_{H=0} (\mu_1 + \mu_2 x^{-4} + \mu_3 x^4) y dx,$
(r14) $I(t) = \int_{H=0} x^{-4}(\mu_1 + \mu_2 x^4 + \mu_3 x^{-4}) y dx,$
$H = \frac{1}{2}y^2 + \frac{1}{12} + \frac{1}{4}x^4 - tx^3, \;\; t \in (\frac{1}{3}, \infty)$

$$\begin{aligned}
&(\mathrm{r}15) \quad I(t)=\int_{H=0} x^{2}(\mu_1+\mu_2 x^{-4}+\mu_3 x^{4}) y dx,\\
&(\mathrm{r}16) \quad I(t)=\int_{H=0} x^{-2}(\mu_1+\mu_2 x^{4}+\mu_3 x^{-4}) y dx,\\
&\qquad\quad H=\tfrac{1}{2}y^2+\tfrac{1}{4}+\tfrac{1}{12}x^4-tx, \quad t\in(\tfrac{1}{3},\infty)\\
&(\mathrm{r}17) \quad I(t)=\int_{H=0} x^{-5}(\mu_1+\mu_2 x^{2}+\mu_3 x^{-2}) y dx,\\
&(\mathrm{r}18) \quad I(t)=\int_{H=0} x^{-3}(\mu_1+\mu_2 x^{-2}+\mu_3 x^{2}) y dx,\\
&\qquad\quad H=\tfrac{1}{2}y^2+\tfrac{1}{6}-\tfrac{1}{2}x^2-tx^3, \ \ t\in(-\tfrac{1}{3},0).
\end{aligned}$$

One can formulate the following conjecture concerning the maximal number of zeroes of the generating function $I(t)$ and the corresponding maximal number of limit cycles produced by the period annulus (called its *cyclicity*).

Conjecture 1. (cf. [14]) *The period annulus around the center at the origin in* (r0)-(r22) *has the following cyclicity under small quadratic perturbations: three for cases* (r1) *with* $a^* < a < 4$*,* (r3) *with* $\frac{7}{3} < a < 4$*,* (r4) *with* $4 < a < 5$*,* (r5) *with* $a = 4$*,* (r6) *with* $a > 4$ *and* (r10)*, and two otherwise.*

We note that $a^* \in (\frac{5}{3}, 3)$ is determined from a transcendental equation [15] and can be calculated numerically, $a^* = 2.0655...$

Let us mention that for some of the cases Conjecture 1 has already been verified. The standard elliptic case (r0) is studied entirely in [19]. The reversible Hamiltonian case (r2) has been investigated in a series of papers, see [4] and the references therein. Cases (r5) with $b = 2$ and (r10) were considered in [14] and [13], respectively. Cases (r5) for $b \neq 2, \frac{1}{2}$ and (r6) with $b \in (\frac{1}{2}, 2)$ are studied in [2]. Case (r1) with $b \in (-1, \frac{1}{3})$ and $b \in (\frac{1}{3}, 3)$ are considered in [22] and [15], respectively. Below we will concentrate our efforts mainly on the cases where Conjecture 1 remains open and will include in our lists that follow only these unsolved cases.

In order to reduce the number of Hamiltonians, we replace

$(x, y) \to (1/x, y/x^2)$ in cases (r5) and (r6),
$(x, y) \to (x/(-4t), y/(-4t)^{3/2})$ in cases (r7) and (r8),
$(x, y) \to (2tx, (2t)^{3/2}y)$ in case (r9),
$(x, y, t) \to ((-6s)^{1/3}x, y, (-48s)^{-1/3})$ in cases (r11) and (r12),
$(x, y) \to (3tx, 9t^2y)$ in cases (r13) and (r14),
$(x, y) \to (1/(3tx), y/x^2)$ in cases (r15) and (r16),
$(x, y) \to (-x/3t, -y/3t)$ in cases (r17) and (r18).

In this way we obtain a reduced list of cases to study as follows:

$$
\begin{array}{ll}
\text{(r1)} & I(t)=\int_{H=0} x^{-4}(\mu_1+\mu_2 x+\mu_3 x^{-1})y dx, \quad (b\notin(-1,\frac{1}{3})\cup(\frac{1}{3},3))\\
\text{(r11)} & I(t)=\int_{H=0} x(\mu_1+\mu_2 t^{-1}x^{-3}+\mu_3 t x^{3})y dx, \quad (b=\frac{1}{3})\\
\text{(r12)} & I(t)=\int_{H=0} x^{-2}(\mu_1+\mu_2 t x^{3}+\mu_3 t^{-1} x^{-3})y dx, \quad (b=\frac{1}{3})\\
 & H=\frac{1}{2}y^2+\frac{3-b}{6(b+1)}+\frac{b-1}{b+1}x+\frac{1-3b}{2(b+1)}x^2-tx^3
\end{array}
$$

$$
\begin{array}{ll}
\text{(r3)} & I(t)=\int_{H=0} x^{-4}(\mu_1+\mu_2 x^2+\mu_3 x^{-2})y dx,\\
\text{(r4)} & I(t)=\int_{H=0} x^{-2}(\mu_1+\mu_2 x^{-2}+\mu_3 x^{2})y dx,\\
 & H=\frac{1}{2}y^2+\frac{b+3}{24(b+1)}+\frac{b-1}{4(b+1)}x^2+\frac{3b+1}{8(b+1)}x^4-tx^3
\end{array}
$$

$$
\begin{array}{ll}
\text{(r5)} & I(t)=\int_{H=0} x(\mu_1+\mu_2 x^{-1}+\mu_3 x)y dx, \quad (b=\frac{1}{2})\\
\text{(r6)} & I(t)=\int_{H=0}(\mu_1+\mu_2 x+\mu_3 x^{-1})y dx, \quad (b\notin(\frac{1}{2},2))\\
 & H=\frac{1}{2}y^2+\frac{1-2b}{2(b+1)}x^2+\frac{b-1}{b+1}x^3+\frac{2-b}{4(b+1)}x^4-t
\end{array}
$$

$$
\begin{array}{ll}
\text{(r7)} & I(t)=\int_{H=0} x^{-5}(\mu_1+\mu_2 t^{-1}x^{3}+\mu_3 t x^{-3})y dx,\\
\text{(r8)} & I(t)=\int_{H=0} x^{-2}(\mu_1+\mu_2 t x^{-3}+\mu_3 t^{-1} x^{3})y dx,\\
\text{(r13)} & I(t)=\int_{H=0}(\mu_1+\mu_2 t x^{-4}+\mu_3 t^{-1} x^{4})y dx,\\
\text{(r14)} & I(t)=\int_{H=0} x^{-4}(\mu_1+\mu_2 t^{-1}x^{4}+\mu_3 t x^{-4})y dx,\\
\text{(r15)} & I(t)=\int_{H=0} x^{-6}(\mu_1+\mu_2 t^{-1}x^{4}+\mu_3 t x^{-4})y dx,\\
\text{(r16)} & I(t)=\int_{H=0} x^{-2}(\mu_1+\mu_2 t x^{-4}+\mu_3 t^{-1} x^{4})y dx,\\
 & H=\frac{1}{2}y^2-\frac{1}{3}x^3+\frac{1}{4}x^4-t, \quad t\in(-\frac{1}{12},0)
\end{array}
$$

$$
\begin{array}{ll}
\text{(r9)} & I(t)=\int_{H=0}(\mu_1+\mu_2 t x^{-3}+\mu_3 t^{-1} x^{3})y dx,\\
\text{(r17)} & I(t)=\int_{H=0} x^{-5}(\mu_1+\mu_2 t^{-1}x^{2}+\mu_3 t x^{-2})y dx,\\
\text{(r18)} & I(t)=\int_{H=0} x^{-3}(\mu_1+\mu_2 t x^{-2}+\mu_3 t^{-1} x^{2})y dx,\\
 & H=\frac{1}{2}y^2-\frac{1}{2}x^2+\frac{1}{3}x^3-t, \quad t\in(-\frac{1}{6},0).
\end{array}
$$

It is useful to know the dimension of the Picard-Fuchs system satisfied by the basic integrals $I_k(t)=\int_{\delta(t)} x^k y dx$ involved in the formulas above. Let us rewrite the equation $H=0$ into the form $\frac{1}{2}y^2=\mathcal{D}(x,t)$. Then for any $k\in\mathbb{Z}$ one obtains

$$\int_{\delta(t)} x^k\mathcal{D}' y dx=\int_{\delta(t)} x^k y d(\tfrac{1}{2}y^2)=\tfrac{1}{3}\int_{\delta(t)} x^k dy^3=-\tfrac{k}{3}\int_{\delta(t)} x^{k-1}y^3 dx.$$

Therefore

$$\int_{\delta(t)} x^{k-1}\left(x\mathcal{D}'+\frac{2k}{3}\mathcal{D}\right) y dx=0.$$

Taking $\mathcal{D}=A_0+A_1x+A_2x^2+A_3x^3+A_4x^4$, in terms of the basic integrals $I_k(t)$ the last identity is equivalent to

$$(2k+12)A_4I_{k+3}+(2k+9)A_3I_{k+2}+(2k+6)A_2I_{k+1}+(2k+3)A_1I_k+2kA_0I_{k-1}=0. \quad (3.3)$$

Using (3.3) with $k = 0, 1, 2, \ldots$ and with $k = -1, -2, \ldots$ we obtain that

$$I_k = \mathrm{span}(I_0, I_1, I_2), \quad k \geq 3; \quad I_k = \mathrm{span}(I_{-1}, I_0, I_1, I_2), \quad k \leq -2 \quad (A_4 \neq 0).$$

Similarly, if $A_4 = 0$, one obtains

$$I_k = \mathrm{span}(I_0, I_1), \quad k \geq 2; \quad I_k = \mathrm{span}(I_{-1}, I_0, I_1), \quad k \leq -2 \quad (A_4 = 0, A_3 \neq 0),$$

and finally,

$$I_k = \mathrm{span}(I_0), \quad k \geq 1; \quad I_k = \mathrm{span}(I_{-1}, I_0), \quad k \leq -2 \quad (A_4 = A_3 = 0, A_2 \neq 0).$$

Here "span" means in general a polynomial $R[t, t^{-1}]$ span. As a consequence, we can formulate a result about the dimension of the Picard-Fuchs system satisfied by the basic integrals in (r1), (r3)-(r9) and (r11)-(r18).

Proposition 4. *In cases* (r1), (r3)-(r4) *with* $b = -\frac{1}{3}$, (r6) *with* $b = 2$, (r9), (r11)-(r12), (r17)-(r18) *the Picard-Fuchs system is of dimension 3 while in the remaining cases it is of dimension 4.*

To derive the Picard-Fuchs equations, we use (3.3) together with the relations

$$I_k'(t) = \int_{\delta(t)} \frac{x^k \partial_t \mathcal{D}}{y} dx, \quad \int_{\delta(t)} \frac{x^k \mathcal{D}'}{y} dx = -k I_{k-1}(t), \quad \int_{\delta(t)} \frac{x^k \mathcal{D}}{y} dx = \frac{1}{2} I_k(t). \qquad (3.4)$$

We can use (3.4) to further simplify some of the integrals above. Thus we get the final list of reversible cases of genus one yet to study:

(r1), (r12) $\quad I(t) = \int_{H=0} x^{-4}(\mu_1 + \mu_2 x^{-1} + \mu_3 x) y dx, \quad (b \notin (-1, \frac{1}{3}) \cup (\frac{1}{3}, 3))$

(r11) $\quad I(t) = \int_{H=0} x^{-1}(\mu_1 + \mu_2 x^{-1} + \mu_3 x) y dx, \quad (b = \frac{1}{3})$

$H = \frac{1}{2}y^2 + \frac{3-b}{6(b+1)} + \frac{b-1}{b+1}x + \frac{1-3b}{2(b+1)}x^2 - tx^3,$

$t \in \left(-\frac{2b}{3(b+1)}, \frac{2(1-3b)^2}{3(b+1)(3-b)^2}\right), \quad b < -1; \quad t \in \left(-\frac{2b}{3(b+1)}, 0\right), \quad b \geq \frac{1}{3}$

(r3) $\quad I(t) = \int_{H=0} x^{-4}(\mu_1 + \mu_2 x^2 + \mu_3 x^{-2}) y dx,$

(r4) $\quad I(t) = \int_{H=0} x^{-2}(\mu_1 + \mu_2 x^{-2} + \mu_3 x^2) y dx,$

$H = \frac{1}{2}y^2 + \frac{b+3}{24(b+1)} + \frac{b-1}{4(b+1)}x^2 + \frac{3b+1}{8(b+1)}x^4 - tx^3,$

$t \in \left(\frac{2b}{3(b+1)}, -\frac{2}{3(b+1)}\sqrt{-\frac{3b+1}{b+3}}\right), \quad b \in (-3, -\frac{1}{3});$

$t \in \left(\frac{2b}{3(b+1)}, \infty\right)$ otherwise.

(r6) $\quad I(t) = \int_{H=0}(\mu_1 + \mu_2 x^{-1} + \mu_3 x) y dx, \quad (b \notin (\frac{1}{2}, 2))$

$H = \frac{1}{2}y^2 + \frac{1-2b}{2(b+1)}x^2 + \frac{b-1}{b+1}x^3 + \frac{2-b}{4(b+1)}x^4 - t,$

$t \in \left(-\frac{b}{4(b+1)}, 0\right), \quad b \geq 2; \quad t \in \left(-\frac{b}{4(b+1)}, \frac{(1-2b)^3}{4(b+1)(2-b)^3}\right), \quad b \leq \frac{1}{2}$

$$
\begin{array}{ll}
(\mathrm{r7}),(\mathrm{r14}) & I(t)=\int_{H=0} x^{-1}(\mu_1+\mu_2x^{-1}+\mu_3x)ydx,\\
(\mathrm{r8}) & I(t)=\int_{H=0} x^{-1}(\mu_1+\mu_2x^{-1}+\mu_3t^{-1}x^2)ydx,\\
(\mathrm{r13}) & I(t)=\int_{H=0}(\mu_1+\mu_2x^{-1}+\mu_3t^{-1}x^2)ydx,\\
(\mathrm{r15}) & I(t)=\int_{H=0} x^{-3}(\mu_1+\mu_2x^{-1}+\mu_3x)ydx,\\
(\mathrm{r5}),(\mathrm{r16}) & I(t)=\int_{H=0} x(\mu_1+\mu_2x^{-1}+\mu_3x)ydx,\\
 & H=\frac{1}{2}y^2-\frac{1}{3}x^3+\frac{1}{4}x^4-t,\ \ t\in(-\frac{1}{12},0)\\
(\mathrm{r9}) & I(t)=\int_{H=0}(\mu_1+\mu_2x^{-1}+\mu_3t^{-1}x)ydx,\\
(\mathrm{r17}) & I(t)=\int_{H=0} x^{-2}(\mu_1+\mu_2x^{-1}+\mu_3x)ydx,\\
(\mathrm{r18}) & I(t)=\int_{H=0}(\mu_1+\mu_2x^{-1}+\mu_3x)ydx,\\
 & H=\frac{1}{2}y^2-\frac{1}{2}x^2+\frac{1}{3}x^3-t,\ \ t\in(-\frac{1}{6},0).
\end{array}
$$

We point out that the above formulas are related in each case to the period annulus around the center at $(1,0)$ obtained for $t\in(t_c,t_s)$ where t_c is the level corresponding to the center and t_s is the level of the contour at which the annulus terminates.

4. The generating function in the Lotka-Volterra case

As in the previous section, we will obtain the complete list of generating functions $I(t)$ for the generalized Lotka-Volterra systems (2.1) in the cases when all their orbits are elliptic curves. Apart of the reversible case, the equivalence classes for the Lotka-Volterra case listed in Propositions 1 and 2 are a result of *affine* transformations. Therefore we can choose by one representative from each such a class and then use the generating function corresponding to it. We will always choose the cases whose parameters satisfy conditions (I) and (II) above. Take real coordinates $z=x+iy$ and consider a small quadratic perturbation as in (3.1). In the reversible case (2.4), the first integral and integrating factor are respectively [14]

$$
H(X,y)=X^\lambda\left(\frac{y^2}{2}-\frac{\lambda(\lambda-1)^2}{32(\lambda+2)}\left(X-\frac{\lambda+2}{\lambda}\right)^2\right),\quad M=X^{\lambda-1} \tag{4.1}
$$

where

$$
\lambda=\frac{b+1}{b-1},\quad b\neq 0,\pm1,\frac{1}{3},\quad X=1-\frac{4x}{\lambda-1}.
$$

The generating function for small quadratic perturbations reads [14], Theorem 3:

$$
I(t)=\int_{H=t} x^{\lambda-1}[\mu_1y+\mu_2yx^{-1}+\mu_3(x-1)y^{-1}]dx. \tag{4.2}
$$

In the Hamiltonian triangle case (rlv1), the integral $I(t)$ takes another form and this case has already been studied in [12]. For this reason (rlv1) is not included in our list below. By (4.1), (4.2), (2.6) and the formulas of first integrals of genus one (rlv2)–(rlv6), we get the list of explicit expressions of $I(t)$ for all other reversible

Lotka-Volterra cases as follows:

$$\begin{aligned}
&\text{(rlv2)} \quad I(t) = \textstyle\int_{H=t} x^{-4}(\mu_1 y + \mu_2 y x^{-1} + \mu_3 (x-1) y^{-1}) dx, \\
&\qquad\quad\; H = x^{-3}(\tfrac{1}{2}y^2 - (x - \tfrac{1}{3})^2), \quad t \in (-\tfrac{4}{9}, 0) \\
&\text{(rlv3)} \quad I(t) = \textstyle\int_{H=t} x(\mu_1 y + \mu_2 y x^{-1} + \mu_3 (x-1) y^{-1}) dx, \\
&\qquad\quad\; H = x^{2}(\tfrac{1}{2}y^2 - (x - 2)^2), \quad t \in (-1, 0) \\
&\text{(rlv4)} \quad I(t) = \textstyle\int_{H=t} x^{-5}(\mu_1 y + \mu_2 y x^{-1} + \mu_3 (x-1) y^{-1}) dx, \\
&\qquad\quad\; H = x^{-4}(\tfrac{1}{2}y^2 - (x - \tfrac{1}{2})^2), \quad t \in (-\tfrac{1}{4}, 0) \\
&\text{(rlv5)} \quad I(t) = \textstyle\int_{H=t} x^{-\frac{5}{2}}(\mu_1 y + \mu_2 y x^{-1} + \mu_3 (x-1) y^{-1}) dx, \\
&\qquad\quad\; H = x^{-\frac{3}{2}}(\tfrac{1}{2}y^2 + (x + \tfrac{1}{3})^2), \quad t \in (\tfrac{16}{9}, \infty) \\
&\text{(rlv6)} \quad I(t) = \textstyle\int_{H=t} x^{-\frac{3}{2}}(\mu_1 y + \mu_2 y x^{-1} + \mu_3 (x-1) y^{-1}) dx, \\
&\qquad\quad\; H = x^{-\frac{1}{2}}(\tfrac{1}{2}y^2 + (x + 3)^2), \quad t \in (16, \infty).
\end{aligned}$$

Clearly the elliptic curves in cases (rlv5) and (rlv6) are obtained by introducing a new variable $x \to x^2$ so that they are given by the level sets of $H(x^2, y)$. Let us also recall that the functions $I(t)$ in (rlv2)-(rlv6) are the coefficients at ε^2 in the expansion of the displacement map obtained for the special perturbations in (3.1) which keep the coefficient at ε zero.

What concerns the small quadratic perturbations of the generic (nonreversible) Lotka-Volterra system, in (3.1) we have $M = x^{\lambda-1}y^{\mu-1}$ (modules needed outside the first quadrant but we will omit them) where H is determined by (2.2). In the generic case $I(t)$ is given by the complete elliptic integral [14], Theorem 2

$$I(t) = \iint_{Int\ \delta(t)} x^{\lambda-1}y^{\mu-1}(\mu_1 + \mu_2 x^{-1} + \mu_3 y^{-1}) dx dy. \tag{4.3}$$

Using (4.3) and the list of first integrals of genus one (lv1)–(lv5), we easily obtain the concrete form of $I(t)$ for all the cases as follows:

$$\begin{aligned}
&\text{(lv1)} \quad I(t) = \textstyle\int_{H=t} x^{-\frac{1}{3}}y^{\frac{1}{3}}(\mu_1 + \mu_2 x^{-1} + \mu_3 y^{-1}) dx, \\
&\qquad\quad H = x^{\frac{2}{3}}y^{\frac{1}{3}}(1 - x - y), \quad t \in (0, 432^{-1/3}), \\
&\text{(lv2)} \quad I(t) = \textstyle\int_{H=t} x^{-\frac{7}{6}}y^{-\frac{1}{3}}(\mu_1 + \mu_2 x^{-1} + \mu_3 y^{-1}) dx, \\
&\qquad\quad H = x^{-\frac{1}{6}}y^{-\frac{1}{3}}(1 - x - y), \quad t \in (432^{1/6}, \infty), \\
&\text{(lv3)} \quad I(t) = \textstyle\int_{H=t} x^{-\frac{7}{6}}y^{-\frac{1}{2}}(\mu_1 + \mu_2 x^{-1} + \mu_3 y^{-1}) dx, \\
&\qquad\quad H = x^{-\frac{1}{6}}y^{-\frac{1}{2}}(1 - x - y), \quad t \in (432^{1/6}, \infty), \\
&\text{(lv4)} \quad I(t) = \textstyle\int_{H=t} x^{-\frac{5}{4}}y^{-\frac{1}{2}}(\mu_1 + \mu_2 x^{-1} + \mu_3 y^{-1}) dx, \\
&\qquad\quad H = x^{-\frac{1}{4}}y^{-\frac{1}{2}}(1 - x - y), \quad t \in (2\sqrt{2}, \infty), \\
&\text{(lv5)} \quad I(t) = \textstyle\int_{H=t} x^{-\frac{4}{3}}y^{-\frac{1}{2}}(\mu_1 + \mu_2 x^{-1} + \mu_3 y^{-1}) dx, \\
&\qquad\quad H = x^{-\frac{1}{3}}y^{-\frac{1}{2}}(1 - x - y), \quad t \in (432^{1/6}, \infty).
\end{aligned}$$

As in Section 3, we formulate a general conjecture about the cyclicity of the period annulus in the Lotka-Volterra systems having all their orbits elliptic or conic curves.

Conjecture 2. *The cyclicity under small quadratic perturbations of the period annulus surrounding the center at the origin in the Lotka-Volterra system* (2.1) *is as follows: three in case* (rlv1) (*the Hamiltonian triangle*) *and two in all other cases* (rlv2)-(rlv6), (lv1)-(lv5).

Except for the isochronous center $\mathcal{S}_1$ and the Hamiltonian triangle, it is very likely that Conjecture 2 is still open. We recall that quadratic perturbations of the general Lotka-Volterra system have been considered in [23]. However, in the recent book [1], page 379, V.I. Arnold declared that the problem with the Lotka-Volterra system still remains open.

Remark about the codimension four case. In this remark we discuss in brief the generic (non-reversible) codimension 4 center. In complex coordinate $z = x + iy$, the related system becomes

$$\dot{z} = -iz + 4z^2 + 2|z|^2 + \alpha\bar{z}^2, \quad \alpha \in \mathbb{C} \setminus \mathbb{R}, \quad |\alpha| = 2.$$

In $(\bar{x}, \bar{y}) = (X^2, Y)$ coordinates, where

$$Y = (2+b)x + cy, \quad X = 1 + 8x + \frac{4}{2+b}Y^2, \quad \alpha = b + ic,$$

the system has a first integral of the form [14] (below the bars are omitted)

$$H(x,y) = \frac{x^{-3}}{8(2-b)}\left(\frac{4y^3}{3(2+b)} + \frac{4y^2}{2+b} + (1-x^2)y - x^2 + \frac{1}{3}\right).$$

It is seen that the level sets of this first integral are (generically) elliptic curves. Therefore the generating function $I(t)$ whose zeroes correspond to limit cycles in the perturbed system are given by the following complete elliptic integral (cf. [14], Theorem 2 (iii))

$$I(t) = \iint_{H(x,y)<t} x^{-6}[\mu_1 + \mu_2 y + \mu_3 y^3 + \mu_4(\kappa^2 y^4 - x^4)]dxdy, \;\; \kappa = \frac{4}{2+b}.$$

The conjecture that the integral $I(t)$ has at most three zeroes in the interval $(t_c, t_s) = (-\frac{1}{12(2-b)}, 0)$ corresponding to the period annulus around the center at $(x_c, y_c) = (1, 0)$ is still open.

5. Zeros of Abelian integrals for some of the reversible cases

In this section we study the generating functions $I(t)$ related to the reversible quadratic systems (r18) and (r11). These systems contain no parameter, have a unique period annulus around the center at $(1,0)$ and the Picard-Fuchs system satisfied by the components of $J(t) = I'(t)$ is of dimension three and two (respectively). Our main result is the following.

Theorem 3. *The exact upper bound of the number of the limit cycles produced by the period annulus under quadratic perturbations of the reversible system* (r18) *or* (r11) *is two.*

To prove Theorem 3 we study first one-parameter analytic quadratic perturbations of (r18) and (r11). According to the formulas derived in Section 3, the number of the limit cycles of such perturbations is bounded by the number of the zeros of the following generating functions

$$I(t)=\int_{\delta(t)}(\mu_1+\mu_2x^{-1}+\mu_3x)ydx,\ \ t\in\left(-\frac{1}{6},0\right),\ \ H_t=\frac{1}{2}y^2-\frac{1}{2}x^2+\frac{1}{3}x^3-t,\ (5.1)$$

for system (r18) and

$$I(t)=\int_{\delta(t)}x^{-1}(\mu_1+\mu_2x^{-1}+\mu_3x)ydx,\ \ t\in\left(-\frac{1}{6},0\right),\ \ H_t=\frac{1}{2}y^2+\frac{1}{3}-\frac{1}{2}x-tx^3,\ (5.2)$$

for system (r11). In the above integrals $\delta(t)$ denotes the unique oval of the real algebraic curve $\{(x,y)\in\mathbb{R}^2:H_t(x,y)=0\}$ for a given $t\in(-\frac{1}{6},0)$ and μ_i are arbitrary real constants. The ovals $\delta(t)$ in (5.1) form a period annulus which is bounded by the homoclinic loop $\{H_0=0\}$ going through the saddle at the origin, while in the case (5.2) they form a period annulus bounded by the parabola $\{H_0=0\}$. In both cases, the ovals $\delta(t)$ exist for $t\in(-\frac{1}{6},0)$, $I(t)$ is analytic in a neighborhood of $t=-1/6$, and $I(-1/6)=0$. Consider the derivative $J(t)=I'(t)$, see formulae (5.12), (5.13) bellow. The key ingredient of the proof of Theorem 3 is the following result.

Theorem 4. *The three-dimensional vector space of Abelian integrals $J(t)=I'(t)$, $t\in[-\frac{1}{6},0)$, defined by* (5.1) *or* (5.2) *is Chebyshev. This means that each integral $J(t)$ has at most two zeros* (*counted with multiplicity*) *in the interval $[-\frac{1}{6},0)$.*

Before proving Theorem 4 we need some preparation.

5.1. Monodromy of the level curves. Let us denote

$$\Gamma_t=\{(x,y)\in\mathbb{C}^2,\ H_t(x,y)=0\},\quad \overline{\Gamma}_t=\{[x:y:z]\in\mathbb{CP}^2,\ H_t\left(\frac{x}{z},\frac{y}{z}\right)=0\}.$$

If $t\neq-\frac{1}{6},0$, then both Γ_t and $\overline{\Gamma}_t$ are smooth curves and $\overline{\Gamma}_t$ is a compact Riemann surface of genus one which is also the compactification of the affine elliptic curve Γ_t. We have $\overline{\Gamma}_t=\Gamma_t\cup\infty$ where $\infty=[0:1:0]$ and therefore

$$rank\ H_1(\Gamma_t,\mathbb{Z})=2,\quad rank\ H_1(\overline{\Gamma}_t,\mathbb{Z})=2.$$

For (5.2), all compact complex level curves $\overline{\Gamma}_t$ have two complex conjugate common points, namely: $P^+=(0,i\sqrt{\frac{2}{3}})$ and $P^-=(0,-i\sqrt{\frac{2}{3}})$ in affine coordinates, as well as one common point at infinity. For (5.1) there is one common point, the point at infinity. For both cases, when blowing-up these points on the complex projective

plane $\mathbb{P}^2$, one obtains a compact smooth rational surface S and an analytic map $S \xrightarrow{\pi} \mathbb{P}$ where the projection π is induced by one of the rational maps

$$\mathbb{C}^2 \to \mathbb{C} : (x,y) \mapsto t = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3, \tag{5.3}$$

$$\mathbb{C}^2 \dashrightarrow \mathbb{C} : (x,y) \mapsto t = \frac{\frac{1}{2}y^2 + \frac{1}{3} - \frac{1}{2}x}{x^3}. \tag{5.4}$$

In both cases S is an elliptic surface in the sense of Kodaira [16] with three singular fibers $\pi^{-1}(0), \pi^{-1}(-1/6), \pi^{-1}(\infty)$. In particular, this allows one to compute the global monodromy of the related homology bundle as described below.

5.1.1. The monodromy of the fibration (5.3). Denote by $0^{\pm}$ the two points on Γ_t with coordinates $(x = 0, y = \pm\sqrt{2t})$ and let $\widehat{\Gamma}_t = \overline{\Gamma}_t \setminus \{0^+, 0^-\}$. We shall determine the monodromy of the homology bundle associated to (5.3) with fibers $H_1(\widehat{\Gamma}_t, \mathbb{Z}) = \mathbb{Z}^3$. Let $\delta(t)$, $\gamma(t)$ be a continuous family of cycles, vanishing at $t = -1/6$ and $t = 0$ respectively, and let $\alpha(t)$ be a cycle represented by a small loop around 0^+ on $\widehat{\Gamma}_t$. We need to describe the monodromy of these cycles on the plane $\mathbb{C} \setminus \{0, -1/6\}$.

The precise definition of the families $\delta(t)$, $\gamma(t)$ is as follows. For $t \in (-1/6, 0)$ the polynomial $\frac{1}{3}x^3 - \frac{1}{2}x^2 - t$ has three real roots $x_1(t) < 0 < x_2(t) < x_3(t)$. Let l be a simple loop on $\mathbb{C} \setminus \{x_1(t), 0, x_2(t), x_3(t)\}$ which makes one turn about $x_1(t), 0, x_2(t)$ in a positive direction and does not contain in its interior $x_3(t)$.

We define $\gamma(t)$ to be the cycle represented by l on $\widehat{\Gamma}_t$ (that is to say by one of the two pre-images of l under the projection $(x,y) \mapsto x$). The cycle $\gamma(t)$ is defined up to a sign. The cycle $\pm\delta(t)$ is defined in a similar way, but the simple loop l is supposed to make one turn about $x_2(t), x_3(t)$ in a positive direction, and does not contain $x_1(t), 0$ in its interior.

We note that $x_1(0) = x_2(0) = 0$, $x_3(0) > 0$ and hence $\gamma(t)$ is a cycle vanishing at $t = 0$. Similarly, $\delta(t)$ is a vanishing cycle at $t = -1/6$, as $x_1(-1/6) < 0$, $x_2(-1/6) = x_3(-1/6) > 0$.

Let

$$\Pi : \ \pi_1(\mathbb{CP}^1 \setminus \Delta, t_0) \to Aut\,(H_1(\widehat{\Gamma}_t, \mathbb{Z})), \ \ \Delta = \{-\tfrac{1}{6}, 0, \infty\}, \ t_0 \in (-\tfrac{1}{6}, 0)$$

be the monodromy representation related to the elliptic fibration associated to (5.3). The image

$$\Pi(\pi_1(\mathbb{CP}^1 \setminus \Delta, t_0))$$

is a group generated by $l_0^*, l_1^*, l_\infty^* = l_1^* \circ l_0^*$, the monodromy operators corresponding respectively to the oriented loops $l_0, l_1, l_\infty = l_1 \circ l_0$. Here l_0 is a simple loop which makes one turn about 0 in a positive direction and does not contain $-1/6$ in its interior. Similarly l_1 is a simple loop which makes one turn about $-1/6$ in a positive direction and does not contain 0 in its interior.

Lemma 1. *In the global basis* $(\alpha(t), \delta(t), \gamma(t))$ (*with appropriate orientation of the*

cycles), the monodromy operators associated to the fibration defined by (5.3) *are:*

$$l_0^* : M_0 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad l_1^* : M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$l_\infty^* : M_\infty = M_1 M_0 = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Proof. The identity

$$2\pi i \int_{\alpha(t)} \frac{dx}{xy} = \frac{1}{\sqrt{t}}$$

shows that $l_0^*(\alpha(t)) = -\alpha(t)$. The remaining claims follow from the usual Picard-Lefschetz formula (it is enough to describe the monodromy of the roots $x_i(t)$) and therefore the details are omitted. $\square$

5.1.2. The monodromy of the fibration (5.4). Here we determine the monodromy representation

$$\Pi : \pi_1(\mathbb{CP}^1 \setminus \Delta, t_0) \to Aut\,(H_1(\overline{\Gamma}_t, \mathbb{Z})), \ \ \Delta = \{-\tfrac{1}{6}, 0, \infty\}, t_0 \in (-\tfrac{1}{6}, 0)$$

associated to the elliptic fibration defined by (5.4). The automorphisms l_0^*, l_1^*, l_∞^* associated to the oriented loops l_0, l_1, l_∞ are defined as in the preceding subsection.

For (5.4) the global monodromy does not follow from the Picard-Lefschetz formula. The local monodromy (around the singular fibers), however, depends only on the topological type of these fibers and is computed in [16]. The topological type of the fibers $\pi^{-1}(0)$, $\pi^{-1}(-1/6)$, $\pi^{-1}(\infty)$ on its turn is computed in [5] and it is respectively (III), (I_1), (IV^*) (using the Kodaira notations, see e.g. [17, Table 6.1]). We conclude that, up to a conjugation, the monodromy operators are given by (e.g. [10, Table 1])

$$\begin{aligned} l_\infty^* : M_\infty &= \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \quad (\mathrm{IV}^*), \\ l_1^* : M_1 &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad (\mathrm{I}_1), \\ l_0^* : M_0 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (\mathrm{III}). \end{aligned}$$

By abuse of notation, we denote by $\delta(t) \in H_1(\overline{\Gamma}_t, \mathbb{Z})$, $t \in (-1/6, 0)$, the continuous family of cycles, vanishing when $t \to -\frac{1}{6}$ along a path connecting t to $-\frac{1}{6}$ in $(-\frac{1}{6}, 0)$. Let $\gamma(t) = l_0^*(\delta(t))$. The continuous families of cycles $\delta(t), \gamma(t)$ are defined in $\mathbb{C} \setminus \{-\frac{1}{6}, 0\}$. According to [18], $(\delta(t), \gamma(t))$ is a basis of $H_1(\overline{\Gamma}_t, \mathbb{Z})$.

Lemma 2. *In the global basis* $(\delta(t), \gamma(t))$ *(with appropriate orientations of the cycles), the monodromy operators associated to the fibration defined by* (5.4) *are:*

$$l_0^* : M_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; l_1^* : M_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \; l_\infty^* : M_\infty = l_1^* \circ l_0^* = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Proof. For the global sections δ, γ of the homology bundle we have $l_0^* \delta = \gamma$, $l_1^* \delta = \delta$ and hence

$$M_0 = \begin{pmatrix} 0 & * \\ 1 & * \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}.$$

The relations $Tr(M_1) = 2$, $Tr(M_0) = 0$, $det(M_0) = 1$, $Tr(M_1 M_0) = Tr(M_\infty) = -1$ imply the result. □

5.2. Wronskians. Let

$$\omega_k = \frac{x^k dx}{y}, \quad k \in \mathbb{Z}$$

be polynomial one-forms on $\mathbb{C}^2$. They induce meromorphic differential one-forms on the compact Riemann surface $\overline{\Gamma}_t$ which are denoted in the same way.

Let us introduce the Wronskian

$$W_{\delta(t),\gamma(t)}(\omega_k, \omega_0) = \int_{\delta(t)} \omega_k \int_{\gamma(t)} \omega_0 - \int_{\delta(t)} \omega_0 \int_{\gamma(t)} \omega_k$$

where, by abuse of notation, $\delta(t), \gamma(t)$ are continuous families of *closed loops*, with the properties :

- they intersect transversally at a single point
- $\overline{\Gamma}_t \setminus \{\delta(t), \gamma(t)\}$ is homeomorphic to a rectangle whose opposite sides are identified to $\delta(t)$ and $\gamma(t)$
- The poles of ω_k are contained in the interior of this rectangle

As ω_0 is holomorphic in both cases, the above Wronskian can be computed by making use of the reciprocity law for abelian differentials of the second and third kind [7, p. 647] as shown next. Below, we denote by C a certain nonzero constant. In the case (5.2) the Wronskian is a rational function and

$$W_{\delta,\gamma}(\omega_k, \omega_0) = \frac{Res_\infty \omega_k \int_\infty^P \omega_0}{2\pi i} = \begin{cases} \frac{C}{t}, & k = 1, \\ 0, & k = 2, \\ \frac{C}{t^2}, & k = 3. \end{cases} \tag{5.5}$$

In a similar way, in the case (5.1) we have

$$W_{\delta,\gamma}(\omega_{-1},\omega_0)=\frac{Res_{0^-}\omega_{-1}}{2\pi i}\int_{P_0}^{0^-}\omega_0+\frac{Res_{0^+}\omega_{-1}}{2\pi i}\int_{P_0}^{0^+}\omega_0=\frac{1}{\sqrt{2t}}\int_{0^-}^{0^+}\omega_0 \tag{5.6}$$

and

$$W_{\delta,\gamma}(\omega_1,\omega_0)=\frac{Res_\infty\omega_1\int_\infty^P\omega_0}{2\pi i}=C\neq 0. \tag{5.7}$$

5.3. Asymptotics of the Abelian integrals. Here we study, for suitable k, the asymptotical behavior of $J_k(t)=\int_{\delta(t)}\omega_k$, where $\delta(t)$ is the continuous family of cycles vanishing at the center (associated to $t=-1/6$), near $t=\infty$ and $t=0$. Below, we shall write $J(t)\lesssim(t-t_0)^s\log(t-t_0)^r, r,s\in\mathbb{R}$ provided that for every sector S centered at the critical value $t_0\in\Delta$ there exists a nonzero constant C_S such that $|J(t)|\leq C_S|t-t_0|^s|\log(t-t_0)^r|$.

5.3.1. The case (5.1). Here we have

$$J_k(t)=\int_{\delta(t)}\frac{x^k dx}{y}=\int_{\delta(t)}\frac{x^k dx}{\sqrt{-\frac{2}{3}x^3+x^2+2t}},\quad k=0,\pm1.$$

(a) Near $t=\infty$, the change of variables $x=t^{\frac{1}{3}}u$ leads to

$$J_k(t)=t^{\frac{k}{3}-\frac{1}{6}}\int_{\tilde{\delta}(t)}\frac{u^k du}{\sqrt{-\frac{2}{3}u^3+t^{-\frac{1}{3}}u^2+2}}=t^{\frac{k}{3}-\frac{1}{6}}\tilde{J}_k(t).$$

When $|t|\to\infty$ (with bounded argument) the integral $\tilde{J}_k(t)$ tends to a finite constant. Consequently

$$J_k(t)\lesssim t^{\frac{k}{3}-\frac{1}{6}},\quad k\in\mathbb{Z}. \tag{5.8}$$

(b) Near $t=0$, the Abelian integral $J_k(t)$ can be expanded as follows

$$J_k(t)=-\frac{\ln(t)}{2\pi i}\int_{\gamma(t)}\omega_k-\frac{1}{2}\int_{\alpha(t)}\omega_k+Q(t)$$

where Q is a meromorphic function in a neighborhood of $t=0$ (this follows from Lemma 1). Therefore

$$J_k(t)=\frac{C}{\sqrt{t}}+P(t)\log(t)+Q(t)$$

where P,Q are meromorphic in a neighborhood of $t=0$. We claim that

$$J_k(t)\lesssim\begin{cases}\log(t) & \text{if } k=0,\\ 1 & \text{if } k=1,\\ \frac{1}{\sqrt{t}} & \text{if } k=-1.\end{cases} \tag{5.9}$$

Indeed, for $k \geq 1$

$$\lim_{t\to 0^-} J_k(t) = 2\int_0^{x_3(0)} \frac{x^{k-1}dx}{\sqrt{-\frac{2}{3}x+1}}$$

where $x_3(0) = 3/2$. For $k = 0$ the integral $J_0(t)$ is of the first kind and its behavior is well known. For $k = -1$ the leading term of the expansion of $J_{-1}(t)$ is given by

$$\int_{\sqrt{t}}^{\infty} \frac{dx}{x\sqrt{x^2-t}} = \frac{1}{\sqrt{t}}\int_1^{\infty} \frac{dx}{x\sqrt{x^2-1}}.$$

(c) Near $t = -1/6$ the functions $J_k(t)$ are holomorphic and $J_k(t) \lesssim 1$.

5.3.2. The case (5.2). We have

$$J_k(t) = \int_{\delta(t)} \frac{x^k dx}{y} = \int_{\delta(t)} \frac{x^k dx}{\sqrt{2tx^3+x-\frac{2}{3}}}, \quad k = 1, 2, 3.$$

(a) Near $t = \infty$, the change of variables $x = t^{-\frac{1}{3}}u$ leads to

$$J_k(t) = t^{-\frac{k+1}{3}} \int_{\tilde{\delta}(t)} \frac{u^k du}{\sqrt{2u^3+t^{-\frac{1}{3}}u-\frac{2}{3}}} = t^{-\frac{k+1}{3}}\tilde{J}_k(t).$$

When $|t| \to \infty$ (with a bounded argument) the integral $\tilde{J}_k(t)$ tends to a finite constant (possibly zero) and hence

$$J_k(t) \lesssim t^{-\frac{k+1}{3}}, \quad k \in \mathbb{Z}. \tag{5.10}$$

(b) Near $t = 0$ any Abelian integral of the first or second kind is a power series in $t^{1/4}$ (because the eigenvalues of l_0^* are fourth roots of the unity). As t tends to 0 two roots of $tx^3 + x/2 - 1/3$ tend to ∞ which suggests us to make the change of variables $x = t^{-\frac{1}{2}}u$. Then we have

$$J_k(t) = t^{-(\frac{k}{2}+\frac{1}{4})} \int_{\tilde{\delta}(t)} \frac{u^k du}{\sqrt{2u^3+u-\frac{2}{3}t^{\frac{1}{2}}}} = t^{-(\frac{k}{2}+\frac{1}{4})}\tilde{J}_k(t)$$

and hence

$$J_k(t) \lesssim t^{-(\frac{k}{2}+\frac{1}{4})}, \quad k \in \mathbb{Z}. \tag{5.11}$$

(c) Near $t = -1/6$ the integrals $J_k(t)$ are holomorphic and $J_k(t) \lesssim 1$.

5.4. Proof of Theorem 4. We have to prove that the derivative

$$J(t) = I'(t) = \mu_1 \int_{\delta(t)} \frac{x^2 dx}{y} + \mu_2 \int_{\delta(t)} \frac{x dx}{y} + \mu_3 \int_{\delta(t)} \frac{x^3 dx}{y} \tag{5.12}$$

(in the case (r11)), or

$$J(t) = I'(t) = \mu_1 \int_{\delta(t)} \frac{dx}{y} + \mu_2 \int_{\delta(t)} \frac{dx}{xy} + \mu_3 \int_{\delta(t)} \frac{xdx}{y} \tag{5.13}$$

(in the case (r18)) has at most two zeros. For this, we use the method of Petrov, see e.g. [20], based on the argument principle.

Introduce the function

$$F(t) = \frac{J(t)}{J_0(t)}, \quad J_0(t) = \int_{\delta(t)} \frac{dx}{y}.$$

It is real analytic on $(-1/6, 0)$ and has an analytic continuation in the complex domain $\mathcal{D} = \mathbb{C} \setminus [0, \infty)$, because the period $J_0(t)$ of the elliptic curve $\bar{\Gamma}_t$ does not vanish there, including at the point $t = -1/6$. To bound the number of the zeros of F in $\mathcal{D}$ it is enough to find this number in the smaller domain $\mathcal{D}_{Rr} = \mathcal{D} \cap \{t : r < |t| < R\}$ for r sufficiently small and R sufficiently big. We are going to evaluate the increment of the argument of F along the boundary of $\mathcal{D}_{Rr}$, oriented in a positive direction.

5.4.1. Zeros of F in the case (r18) and (5.1). Along the boundary of the small circle $\{|t| = r\}$, according to (5.9), the increase of the argument of F is at worst close to π, and along the boundary of the big circle $\{|t| = R\}$ this increase is at worst close to $2\pi/3$, see (5.8). Denote by $F^{\pm}$ the restriction on $(0, \infty)$ of the analytic function obtained as an analytic continuation of F along an arc contained in the upper (lower) complex half-plane $\mathrm{Im} \pm t > 0$ respectively. The function F is real analytic on $(-\infty, 0)$ which implies that along the interval $(0, \infty)$ the imaginary part $\mathrm{Im}\,(F^+(t))$ of F^+ satisfies

$$2i\, \mathrm{Im}\,(F^+(t)) = F^+(t) - \overline{F^+(t)} = F^+(t) - F^-(t).$$

Assume for a moment that $F^+(t) - F^-(t)$ has at most one simple zero on the interval $(0, \infty)$. Then *summing up the above information we conclude that $F(t)$ has at most two zeros in the complex domain $\mathcal{D}_{Rr}$, and hence in $\mathcal{D}$.* This result is obviously exact.

It remains to prove that $F^+(t) - F^-(t)$ has at most one zero on $(0, \infty)$. Clearly F^- is an analytic continuation of F^+ along an arc making one turn about $t = 0$ in a positive direction. This shows that $F^+(t) - F^-(t)$ is obtained as an analytic continuation of the function

$$F(t) - l_0^*(F(t)) = \frac{\int_{\delta(t)} \omega}{\int_{\delta(t)} \omega_0} - \frac{\int_{l_0^*\delta(t)} \omega}{\int_{l_0^*\delta(t)} \omega_0} = \frac{W_{\delta(t), l_0^*\delta(t)}(\omega, \omega_0)}{\int_{\delta(t)} \omega_0 \int_{l_0^*\delta(t)} \omega_0}, \quad t \in (-1/6, 0)$$

along an arc contained in the upper complex half-plane. Here $\omega = \mu_1\omega_0 + \mu_2\omega_{-1} + \mu_3\omega_1$, $l_0^*\delta(t), \delta(t)$ are cycles on the elliptic curve defined by $H_t = 0$, with two removed points $0^{\pm}$, and $l_0^*\delta(t) = \delta(t) - \gamma(t) + \alpha(t)$. Let

$$iW(t), \quad t \in (0, \infty)$$

be the real analytic function obtained as an analytic continuation of

$$iW_{\delta(t),l_0^*\delta(t)}(\omega,\omega_0), \quad t\in(-1/6,0)$$

along an arc contained in the upper complex half-plane. As

$$F^+(t)-F^-(t)=\frac{W(t)}{|\int_{\delta(t)}\omega_0|^2}$$

then we shall show that $iW(t)$ has at most one zero on $(0,\infty)$. We use once again the argument principle. We shall show that the analytic continuation of $W(t)$ in $\tilde{\mathcal{D}}=\mathbb{C}\setminus(-\infty,0)$ has at most one zero counted with multiplicity. For this purpose we consider the complex domain

$$\tilde{\mathcal{D}}_{Rr}=\tilde{\mathcal{D}}\cap\{t:|t|>r,\ \ |t|<R\}.$$

As before, denote by $W^\pm(t)$ the restriction on $(-\infty,0)$ of the analytic function obtained as an analytic continuation of $W(t), t\in(0,\infty)$ along an arc contained in the upper (lower) complex half-plane $\mathrm{Im}\pm t>0$ respectively. Along the interval $(-1/6,0)$ we have

$$\begin{aligned}
W^+(t)-W^-(t) &= W_{\delta(t),l_0^*\delta(t)}(\omega,\omega_0)-(l_0^*)^{-1}W_{\delta(t),l_0^*\delta(t)}(\omega,\omega_0)\\
&= W_{\delta(t),l_0^*\delta(t)}(\omega,\omega_0)-W_{(l_0^*)^{-1}\delta(t),\delta(t)}(\omega,\omega_0)\\
&= W_{\delta(t),\delta(t)+\alpha(t)-\gamma(t)}(\omega,\omega_0)-W_{\delta(t)+\alpha(t)+\gamma(t),\delta(t)}(\omega,\omega_0)\\
&= 2W_{\delta(t),\alpha(t)}(\omega,\omega_0)\\
&= -\frac{2}{\sqrt{2t}}\int_{\delta(t)}\omega_0.
\end{aligned}$$

Note that, according to section 5.1.1., the functions $W^\pm(t)$ are single-valued in a neighborhood of $t=-1/6$. Therefore

$$W^+(t)-W^-(t)=-\frac{2}{\sqrt{2t}}\int_{\delta(t)}\omega_0$$

along the interval $(-\infty,0)$. We conclude that the imaginary part $i(W^+(t)-W^-(t)), t\in(-\infty,0)$ of the analytic continuation of $iW(t), t\in(0,\infty)$ does not vanish.

It follows from the asymptotic expansion of J_k near $t=0$ obtained in 5.3.1.(b) that

$$W(t)\lesssim\frac{1}{\sqrt{t}}, \quad t\sim 0.$$

Therefore along the small circle $\{|t|=r\}$, the increase of the argument of $W(t)$ is at worst close to π. Along the border of the big circle the decrease of the argument is close to $2\pi/6$, see (5.8). Summing up the above information we conclude that the total increase of the argument of $W(t)$ along the border of $\tilde{\mathcal{D}}_{Rr}$ is close to (or smaller than) $4\pi-2\pi/6$ and hence $W(t)$ has at most one zero. *Therefore the maximal number of the zeros of $F(t)$ in $\mathcal{D}$ is two (and this result is exact).*

5.4.2. Zeros of F in the case (r11) and (5.2). In the same way as above we may study $F(t)$ in the case (r11). This leads, however, to a bound of the number of the zeros in $[-1/6, 0)$ equal to three. To improve the bound we consider first the function $\tilde{F}(t) = F(t) + \mu_4$, where μ_4 is a real constant. We shall show that $\tilde{F}(t)$ has at most three zeros in the complex domain $\mathcal{D} = \mathbb{C} \setminus [0, \infty)$. For this we consider once again the domain $\mathcal{D}_{Rr} = \mathcal{D} \cap \{t : r < |t| < R\}$ and evaluate the increase of the argument of $\tilde{F}$ along its border. Along the boundary of the small circle $\{|t| = r\}$, according to (5.11), the increase of the argument of $\tilde{F}(t)$ is close to 3π, and along the boundary of the big circle $\{|t| = R\}$ this increase is close to 0, see (5.10). Along the interval $(0, \infty)$ the imaginary part of F is equal to

$$\frac{W(t)}{|\int_{\delta(t)} \omega_0|^2}$$

where, according to (5.5), for suitable constants c_1, c_2,

$$W(t) = \frac{c_1 t + c_2}{t^2}.$$

It follows that the imaginary part of $F(t)$ has at most one zero along $(0, \infty)$. The argument principle implies that $\tilde{F}(t)$ has at most three zeros in the complex domain $\mathcal{D}$. Suppose now that for some μ_1, μ_2, μ_3 the function $F(t)$ has exactly three zeros in $\mathcal{D}$, and hence in the domain $\mathcal{D}_{Rr}$ for r sufficiently small and R big enough. According to (5.10) for $|t|$ sufficiently big we have

$$F(t) = ct^{-k/3} + o(|t^{-k/3}|).$$

It follows that for sufficiently small μ_4 a real zero of $\tilde{F}(t)$ bifurcates from ∞ on the interval $(-\infty, 0)$ and this zero is not contained in the domain $\mathcal{D}_{Rr}$. Therefore $\tilde{F}(t)$ will have at least four zeros in the domain $\mathcal{D}$, in contradiction with the result proved above. *Therefore the maximal number of the zeros of $F(t)$ in $\mathcal{D}$ is two (and this result is exact).*

Theorem 4 is proved. $\square$

5.5. Proof of Theorem 3. Denote the open period annulus of the fixed reversible system (r18) or (r11) by Π. Let X_λ, $\lambda \in \Lambda$ be the set of all quadratic plane vector fields, analytic with respect to λ and such that X_0 coincides with (r18) (respectively, with (r11)). Theorem 3 can be reformulated as follows:

The cyclicity $Cycl(\Pi, X_\lambda)$ of the open period annulus Π is equal to two

which means that X_λ has *at most two* limit cycles which tend to Π as λ tends to zero. It is shown in [9, Theorem 1] that if the cyclicity $Cycl(\Pi, X_\lambda)$ is finite, then there exists a germ of an analytic curve $\lambda(\varepsilon)$, such that

$$Cycl(\Pi, X_\lambda) = Cycl(\Pi, X_{\lambda(\varepsilon)}).$$

In other words, it is enough to study one-parameter deformations. Let $I(t)$ be the first non-zero Poincaré-Pontryagin (or generating) function associated to such a perturbation. The derivative $J(t) = I'(t)$ is of the form (5.12) or (5.13) and we proved that this function has at most two zeros (Theorem 4), and hence $I(t)$ has at most three zeros on $[-1/6, 0)$, one of them being $t = -1/6$. Therefore, by a standard argument, the cyclicity of the open period annulus of the perturbed one-parameter quadratic system is at most two.

It remains to show that the cyclicity $Cycl(\Pi, X_\lambda)$ is finite. It is shown in [9] that if $Cycl(\Pi, X_\lambda) = \infty$, then there exists a Poincaré-Pontryagin function $I(t)$ (associated to some one-parameter deformation) with infinite number of zeros. As $I(t)$ is an Abelian integral, then this is clearly impossible. This completes the proof of Theorem 3. $\square$

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